一类一阶双曲型偏微分方程的边界控制*

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【摘　要】本文研究了一类一阶双曲型偏微分方程的边界控制问题,利用边界控制的反推法设计出反馈控制器。在设计控制器的过程中,改进了反推法中常用的积分变换,同时经过一系列数学计算解出积分变换中的核函数,从而设计出闭环系统的反馈控制器,最后,为了得到闭环系统的稳定性,找到积分变换的逆变换。

【关键词】一阶双曲型方程;边界控制;反推法;稳定性

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引言

生活中常见的现象,例如弦的振动,波的传播,梁的振动等,常用二阶双曲型方程来描述,而交通流,化学反应器,热交换器等,则常用一阶双曲型偏微分方程来描述。近年来,边界控制的反推法应用在抛物型偏微分方程和二阶双曲型偏微分方程的控制设计中,并取得不少成果[1-4],同时该方法也被应用到一阶双曲型方程的边界控制中[5],本文就利用边界控制的反推法研究了一类一阶双曲型方程。

1 问题陈述

考虑一阶双曲型偏微分方程所组成的控制系统:

$$(1)\quad \begin{aligned} &u_t(x,t)=u_x(x,t)+\lambda u(x_0,t)\\ &u(1,t)=U(t)\end{aligned}$$

其中,$u(x,t)$是状态变量,$U(t)$是控制输入,$\lambda>0$是常数。在系统(1)中,若不加控制即$U(t)=0$时,系统(1)是不稳定的。因此,本文控制设计的目的是找到反馈控制器$U(t)$,使控制系统(1)在控制器$U(t)$下是稳定的。

2 控制的设计

为运用反推法设计控制器,首先,变换

$$w(x,t)=u(x,t)-\int_0^x k(x,y)u(y,t)dy-\int_0^{x_0} r(x,y)u(y,t)dy \tag{2}$$

将控制系统(1)转化为稳定的目标系统(见[5]):

$$\begin{aligned} &w_t(x,t)=w_x(x,t)\\ &w(1,t)=0\end{aligned} \tag{3}$$

其中,$k(x,y)$和$r(x,y)$是待定的核函数。

其次,在变换(2)中取$x=1$,即可得到反馈控制器

$$U(t)=u(1,t)=\int_0^1 k(1,y)u(y,t)dy+\int_0^{x_0} r(1,y)u(y,t)dy. \tag{4}$$

最后,需找到变换(2)的逆变换,证明系统(1)在控制器(4)下是稳定的。

3 核函数的求解

为了得到变换(2)中的核函数$k(x,y)$和$r(x,y)$。首先,在(2)式两边关于x求偏导,有

$$w_x(x,t)=u_x(x,t)-k(x,x)u(x,t)-\int_0^x k_x(x,y)u(y,t)dy-\int_0^{x_0} r_x(x,y)u(y,t)dy. \tag{5}$$

类似地,在变换(2)两边同时关于t求偏导,并由控制系统(1),有

$$\begin{aligned} w_t(x,t)&=u_t(x,t)-\int_0^x k(x,y)u_t(y,t)dy-\int_0^{x_0} r(x,y)u_t(y,t)dy\\ &=u_x+\lambda u(x_0,t)-\int_0^x k(x,y)(u_y(y,t)+\lambda u(x_0,t))dy\\ &\quad-\int_0^{x_0} r(x,y)(u_y(y,t)+\lambda u(x_0,t))dy\\ &=u_x-k(x,x)u(x,t)+(k(x,0)+r(x,0))u(0,t)\\ &\quad+(\lambda-r(x,x_0)-\lambda\int_0^x k(x,y)dy-\lambda\int_0^{x_0} r(x,y)dy)u(x_0,t)\\ &\quad+\int_0^x k_y(x,y)u(y,t)dy+\int_0^{x_0} r_y(x,y)u(y,t)dy\end{aligned} \tag{6}$$

由(5)、(6)可得

$$\begin{aligned} &w_t(x,t)-w_x(x,t)\\ &=(k(x,0)+r(x,0))u(0,t)\\ &\quad+(\lambda-r(x,x_0)-\lambda\int_0^x k(x,y)dy-\lambda\int_0^{x_0} r(x,y)dy)u(x_0,t)\\ &\quad+\int_0^{x_0}(r_x(x,y)+r_y(x,y))u(y,t)dy+\int_0^x(k_x(x,y)+k_y(x,y))u(y,t)dy\end{aligned} \tag{7}$$

为满足目标系统中的方程$w_t(x,t)=w_x(x,t)$,选择核函数$k(x,y)$和$r(x,y)$满足方程组

$$\begin{cases} k_x(x,y)+k_y(x,y)=0\\ r_x(x,y)+r_y(x,y)=0\\ k(x,0)+r(x,0)=0\end{cases} \tag{8}$$

及相容性条件

$$\lambda-r(x,x_0)-\lambda\int_0^x k(x,y)dy-\lambda\int_0^{x_0} r(x,y)dy=0. \tag{9}$$

由$k_x(x,y)+k_y(x,y)=0$可知,核函数$k(x,y)$的解具有$\varphi(x-y)$的形式,故可设$k(x,y)=\varphi(x-y)$。其次,

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运用偏微分方程的变量分离法解核函数 $r(x,y)$,设 $r(x,y)=p(x)q(y)$,则由(8)可得 $p(x)$ 和 $q(y)$ 满足方程

$$\frac{p'(x)}{p(x)}=-\frac{q'(y)}{q(y)}=a \quad (10)$$

其中, a 是待定的常数。

解方程(10)可得

$$p(x)=be^{ax},\ q(y)=ce^{-ay} \quad (11)$$

其中, b,c 是待定的常数。从而得到

$$r(x,y)=p(x)q(y)=bce^{a(x-y)}. \quad (12)$$

又由(6)有 $k(x,0)=-r(x,0)=\varphi(x)=-bce^{ax}$,可得

$$k(x,y)=\varphi(x-y)=-bce^{a(x-y)}. \quad (13)$$

最后,通过验证相容性条件(9),得到 满足的条件。由(9)、(12)和(13)得

$$\begin{aligned}&\lambda-r(x,x_0)-\lambda\int_0^x k(x,y)dy-\lambda\int_0^{x_0} r(x,y)dy\\ &=\lambda-bce^{a(x-x_0)}+\lambda\int_0^x bce^{a(x-y)}dy-\lambda\int_0^{x_0} bce^{a(x-y)}dy\\ &=\lambda-bce^{a(x-x_0)}+\frac{\lambda bc}{a}(e^{ax}-1)-\frac{\lambda bc}{a}(e^{ax}-e^{a(x-x_0)})\\ &=\frac{\lambda}{a}(a-bc)+\frac{bc}{a}(\lambda-a)e^{a(x-x_0)}\\ &=0.\end{aligned}$$

取 $a=bc=\lambda$,使得上式成立,即相容性条件(9)得到满足。方程组(8)的解为

$$r(x,y)=\lambda e^{\lambda(x-y)},\ k(x,y)=-\lambda e^{\lambda(x-y)}. \quad (14)$$

从而由(14)可得变换(2)为

$$\begin{aligned}w(x,t)&=u(x,t)-\int_0^x k(x,y)u(y,t)dy-\int_0^{x_0} r(x,y)u(y,t)dy\\ &=u(x,t)+\lambda\int_0^x e^{\lambda(x-y)}u(y,t)dy-\lambda\int_0^{x_0} e^{\lambda(x-y)}u(y,t)dy.\end{aligned} \quad (15)$$

由(4)、(14)可得控制系统(1)的反馈控制器

$$\begin{aligned}U(t)&=\int_0^1 k(1,y)u(y,t)dy+\int_0^{x_0} r(1,y)u(y,t)dy\\ &=-\lambda\int_0^1 e^{\lambda(1-y)}u(y,t)dy+\lambda\int_0^{x_0} e^{\lambda(1-y)}u(y,t)dy.\end{aligned} \quad (16)$$

4 逆变换

为证明控制系统(1)在控制器(16)下是稳定的,需找到变换(2)的逆变换,事实上,变换(2)是可逆变换,且逆变换具有如下形式

$$u(x,t)=w(x,t)+\int_0^x l(x,y)w(y,t)dy+\int_0^{x_0} h(x,y)w(y,t)dy. \quad (17)$$

变换(17)将目标系统(3)转化为控制系统(1),运用第4节类似的计算方法,可得

$$\begin{aligned}&u_t(x,t)-u_x(x,t)-\lambda u(x_0,t)\\ &=(h(x,x_0)-\lambda)w(x_0,t)-(l(x,0)+h(x,0))w(0,t)\\ &-\int_0^x(l_x(x,y)+l_y(x,y))w(y,t)dy\\ &-\int_0^{x_0}(h_x(x,y)+h_y(x,y)+\lambda l(x_0,y)+\lambda h(x_0,y))w(y,t)dy\end{aligned} \quad (18)$$

为了控制系统中的方程 $u_t(x,t)=u_x(x,t)+\lambda u(x_0,t)$,则核函数 $l(x,y),h(x,y)$ 满足如下方程组

$$\begin{cases}l_x(x,y)+l_y(x,y)=0\\ h_x(x,y)+h_y(x,y)+\lambda l(x_0,y)+\lambda h(x_0,y)=0\\ l(x,0)+h(x,0)=0\\ h(x,x_0)-\lambda=0\end{cases} \quad (19)$$

类似地,运用第4节里面变量分离的方法可解得 $l(x,y)=-\lambda,\ h(x,y)=\lambda$ 。从而变换(2)的逆变换为

$$u(x,t)=w(x,t)-\lambda\int_0^x w(y,t)dy+\lambda\int_0^{x_0} w(y,t)dy. \quad (20)$$

最后,根据变换(15)和逆变换(20)的有界性,可以证明得到控制系统(1)在控制器(19)下是稳定的[6]。

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Boundary Control of a First-order Hyperbolic PDE

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Abstract: This paper considers stabilization of a first-order hyperbolic PDE by using backstepping method of boundary control. For designing the controller, a modificative backstepping transformation is introduced. Through a series of mathematical computation, the exact solutions of kernels are obtained, and a control law is obtained specifically. Finally, for obtaining stabilization of the closed-loop system, the inverse imformation is found.

Key words: a First-order Hyperbolic PDE; boundary control; backstepping method; stabilization

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An Evaluation on Land Utilization Ecological Carrying Capacity Based on the Ecological Footprint Model

——Taking Xichang College as an Example

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Abstract: Based on the ecological footprint mode, the authors calculated and analyzed the land utilization ecological carrying capacity on Xichang city of Sichuan Province. The results showed that the per capita ecological footprint of Xichang City was 0.695541, which dropped to 0.610628 after deducting 12% biodiversity, and the per capita ecological deficit was 0.08491. The utilization level of resources per capita of Xichang City was much lower than that of China. From the compostion of ecological footprint, the ecological footprint of different land use types from big to small order, followed by arable land, fossil energy land, water area, construction land, grassland and woodland. At present, the development mode of Xichang city was mainly to consume natural resource stock.

Key words: ecological footprint; ecological carrying capacity; Xichang city; ecological environment

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variety of methods.L 'Hospital principle is one of the important methods. Based on the confucian classics section Paodingjieniu of Chuang-tzu, the paper expounds in detail how to utilize L' Hospital principle for limit and the problems that should be paid attention to when using it, and the problems explained by the example.

Key words: limit of function; L 'Hospital principle; higher mathematics; Paodingjieniu